Sensitivity Analysis of Linear Programming (LP)

- In all LP models the coefficient of the objective function and the constraints are supplied as input data or as parameters to the model.
- The optimal solutions obtained is based on the values of these coefficients.
- In practice the values of these coefficients are seldom known with absolute certainty.
- Hence the solution of a practical problem is not complete with the mere determination of the optimal solution.
- Each variation in the values of the data coefficients changes the LP problem which may in turn affect the optimal solution found earlier.
- Sensitivity analysis helps to study how the optimal solution will change with changes in the input coefficients.

Example

A factory manufactures three products, which require three resources – labour, material, and administration. The unit profits on these products are Rs. 100, Rs. 60, and Rs. 40 respectively. There are 100 hr of labour, 600 kg of material and 300 hr of administration available per day. The resource requirements for the products to manufacture are given in the table below.

<table>
<thead>
<tr>
<th>Resource</th>
<th>Product 1</th>
<th>Product 2</th>
<th>Product 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Labour</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Material</td>
<td>10</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Administration</td>
<td>2</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

Determine the optimal product mix and do sensitivity analysis.

LINGO Model

Model:
Title Product mix Problem;
SETS:
product/1..3/:p,x;
resource/1..3/:reso;
material(resource,product):a;
endsets
[objective]max=@sum(product(j):p(j)*x(j));
@for(resource(i): [Resource_constraints]
  @sum(product(j):a(i,j)*x(j))<=reso(i));
Data:
p=100, 60, 40;
reso=100, 600, 300;
a= 1 1 1
  10 4 5
  2 2 6;
enddata
end
Solution

Objective value: 7333.333

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>X( 1)</td>
<td>33.33333</td>
<td>0.0000000</td>
</tr>
<tr>
<td>X( 2)</td>
<td>66.66667</td>
<td>0.0000000</td>
</tr>
<tr>
<td>X( 3)</td>
<td>0.0000000</td>
<td>26.66667</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Row</th>
<th>Slack or Surplus</th>
<th>Dual Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>RESOURCE_CONSTRAINTS (1)</td>
<td>0.0000000</td>
<td>33.33333</td>
</tr>
<tr>
<td>RESOURCE_CONSTRAINTS (2)</td>
<td>0.0000000</td>
<td>6.666667</td>
</tr>
<tr>
<td>RESOURCE_CONSTRAINTS (3)</td>
<td>100.00000</td>
<td>0.0000000</td>
</tr>
</tbody>
</table>

- The above solution report is a selected portions of the solution report generated when clicking the ‘solve’ icon in the toolbar of LINGO model window.

- Reduced cost, Dual price, Objective coefficient ranges, and Right hand side ranges are required for sensitivity analysis.

- Objective coefficient ranges, and Right hand side ranges are obtained when the following actions are carried out as the range computations are not enabled by default.
  - Go to the LINGO model window. From the ‘LINGO’ menu of the toolbar select ‘options’. Then select ‘General Solver’ tab and go to ‘Dual Computations’ list box. Select ‘Prices & Ranges’ and click OK and then go to ‘LINGO’ menu and select ‘Range’.

Ranges in which the basis is unchanged:

Objective Coefficient Ranges

<table>
<thead>
<tr>
<th>Variable</th>
<th>Current Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>X( 1)</td>
<td>100.0000</td>
<td>50.0000</td>
<td>40.0000</td>
</tr>
<tr>
<td>X( 2)</td>
<td>60.0000</td>
<td>40.0000</td>
<td>20.0000</td>
</tr>
<tr>
<td>X( 3)</td>
<td>40.0000</td>
<td>26.66667</td>
<td>INFINITY</td>
</tr>
</tbody>
</table>

Right hand Side Ranges

<table>
<thead>
<tr>
<th>Row</th>
<th>Current</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>RHS</td>
<td>100.0000</td>
<td>50.0000</td>
<td>40.0000</td>
</tr>
<tr>
<td>RESOURCE_CONSTRAINTS (1)</td>
<td>600.0000</td>
<td>400.0000</td>
<td>200.0000</td>
</tr>
<tr>
<td>RESOURCE_CONSTRAINTS (3)</td>
<td>300.0000</td>
<td>INFINITY</td>
<td>100.0000</td>
</tr>
</tbody>
</table>

Terms used in the solution reports

Slack or Surplus

- Tells how close you are to satisfying a constraint as equality.

- This quantity, on less-than-or-equal-to (≤) constraints, is generally referred to as slack. On greater-than-or-equal-to (≥) constraints, this quantity is called a surplus.
• If a constraint is exactly satisfied as equality, the slack or surplus value will be zero.

• If a constraint is violated, as in an infeasible solution, the slack or surplus value will be negative.

• Knowing this can help you find the violated constraints in an infeasible model—a model for which there doesn’t exist a set of variable values that simultaneously satisfies all constraints.

**Dual Price (Shadow Price)**

• Dual price is the amount that the objective would improve as the Right-Hand Side (RHS), or constant term, of the constraint is increased by one unit

• In a maximization problem, improve means the objective value would increase.

• In a minimization problem, the objective value would decrease if you were to increase the right-hand side of a constraint with a positive dual price.

• Dual prices are sometimes called shadow prices, because they tell you how much you should be willing to pay for additional units of a resource.

**Reduced Cost (Opportunity Cost)**

• There are two valid, equivalent interpretations of a reduced cost.

• First, you may interpret a variable’s reduced cost as the amount that the objective coefficient of the variable would have to improve before it would become profitable to give the variable in question a positive value in the optimal solution.

• For example, if a variable had a reduced cost of 10, the objective coefficient of that variable would have to increase by 10 units in a maximization problem and/or decrease by 10 units in a minimization problem for the variable to become an attractive alternative to enter into the solution.

• Second, the reduced cost of a variable may be interpreted as the amount of penalty you would have to pay to introduce one unit of that variable into the solution.

• Again, if you have a variable with a reduced cost of 10, you would have to pay a penalty of 10 units to introduce the variable into the solution. In other words, the objective value would fall by 10 units in a maximization model or increase by 10 units in a minimization model.

**Range Report**

• A range report shows over what ranges you can:

  1) Change a coefficient in the objective without causing any of the optimal values of the decision variables to change and

  2) Change a row's constant term (also referred to as the right-hand side coefficient) without causing any of the optimal values of the dual prices or reduced costs to change
**Ranges are valid only if you are planning to alter a single objective coefficient or right-hand side.**

**The range information provided by LINGO cannot be applied in situations where one is simultaneously varying two or more coefficients.**

**You can change a coefficient by any amount up to the amount that is indicated in the range report without causing a change in the optimal solution.**

**Whether the optimal solution will actually change if you exceed the allowable limit is not certain.**

**Interpretation of the information contained in the above solution reports**

**Range analysis on resource constraints**

- The dual price gives the net impact in the maximum profit if additional units of certain resources can be obtained.
- Labour has the maximum impact, providing 33.3 increase in profit for each additional hour of labour.
- The dual prices on the resources apply as long as their variations stay within the prescribed ranges on RHS constants given in the range report of the solution.
- In other words, a Rs. 33.3 increase in profit is achievable for each additional labour hour as long as the increase in labour hour goes up to 150. Similarly, profit decreases by a Rs. 33.3 for each decrease in unit labour hour as long as it is not decreased beyond 60 hrs.

**Range analysis on objective function coefficients**

- The range on the objective function coefficients exhibit the sensitivity of the optimal solution with respect to changes in the unit profits of the three products.
- The optimal solution will not be affected as long as the unit profit of product 1 stays between Rs. 60 and Rs. 150. Similarly, the range given for product 2 can be interpreted.
- Of course, the maximum profit will be affected by the change. For example, if the unit profit on product 1 increases from Rs. 100 to Rs. 120, the optimal solution will be same but the maximum profit will increase to \( (7333.3 + (120-100)X(1)) = 7333.3 + 20 * 33.33 = Rs. 7999.9 \)
- Note that product 3 is not economical to produce. Its opportunity cost measures the negative impact of producing product 3 to the maximum profit.
- Also, the unit profit on product 3 must increase to 66.7 (present value + opportunity cost) before it becomes economical to produce.